Wolff's inequality for hypersurfaces

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Abstract

We extend Wolff's "local smoothing" inequality [18] to a wider class of not necessarily conical hypersurfaces of codimension 1. This class includes surfaces with nonvanishing curvature, as well as certain surfaces with more than one flat direction. An immediate consequence is the L^p-boundedness of the corresponding Fourier multiplier operators.

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The purpose of this article is to extend the "local smoothing" inequality, proved in [18], [5] for circular cones in $\mathbb{R}^d$, $d \geq 2$, and in [9] for more general conical surfaces in $\mathbb{R}^3$, to a wider class of bounded surfaces of codimension 1 in $\mathbb{R}^{d+1}$, $d \geq 3$.

Recall that Wolff's inequality [18] states that if f is a function with Fourier transform supported in a δ-neighbourhood of the segment of the circular cone with $1 \leq |x| \leq 2$, then

$$\forall \epsilon \, \exists C_\epsilon : \ \|f\|_p \leq C_\epsilon \delta^{-\frac{d-1}{2}+\frac{d}{p}-\epsilon} \|f\|_{p,\delta}, \tag{1}$$

with $d = 2$ and $p > 74$; this was then extended in [5] to $d \geq 3$, $p > 2 + \frac{32}{3d-7}$, and $d \geq 4$, $p > 2 + \frac{8}{d-3}$. Here the norm on the right side is defined by

$$\|f\|_{p,\delta} = \left(\sum_a \|\Xi_a * f\|_p^p \right)^{1/p},$$

where $\widehat{\Xi_a}$ are, roughly, cutoffs in the Fourier space corresponding to the natural covering of the δ-neighbourhood of the cone by rectangular "plates"

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of thickness δ. We propose to extend this inequality in two directions. First, we will consider cones generated by more general surfaces of codimension 2. Second, we will also allow surfaces with more than one flat direction, satisfying certain geometrical conditions stated below; the simplest nontrivial examples are the "k-cones", defined below and described in more detail in Section 7. An immediate consequence is the L^p boundedness of Bochner-Riesz multipliers for the same classes of surfaces and for appropriate ranges of exponents.

We now define specific classes of surfaces of interest to us. We will always assume that S is a bounded surface of codimension 1 in $\mathbb{R}^{d+1}$, smooth everywhere except for the possible boundary, with all curvatures bounded from above uniformly by a constant.

A *nondegenerate surface* in $\mathbb{R}^{l+1}$ will be a surface S_0 defined by an equation of the type $x_{l+1} = F(x_1, \ldots, x_l)$, where F is smooth with all derivatives bounded uniformly in x, and all principal curvatures are bounded away from zero:

$$|\langle u, (D_x^2 F)u \rangle| \geq c_0 |u|^2, \ u \in \mathbb{R}^{l+1},$$

with the lower bound c_0 uniform in u. Elliptic surfaces, such as e.g. a sphere or a paraboloid, are clearly nondegenerate. However, we do not require the Hessian to be positive definite, hence we also allow surfaces with both positive and negative curvatures (e.g. hyperboloids).

A *conical surface* in $\mathbb{R}^{d+1}$ is defined as

$$S = \{tx : \ x \in S_0, \ t \in [C_1, C_2]\}, \ 0 < C_1 < C_2 < \infty, \tag{2}$$

where S_0 is a surface of dimension $l = d - 1$ contained in an affine subspace $X \subset \mathbb{R}^{d+1}$ of dimension d which does not pass through the origin, such that S_0 viewed as a subset of X is a nondegenerate surface if X is identified with $\mathbb{R}^d$ in the obvious way. This class includes circular cones as well as more general homogeneous quadrics of the form $S = \{x : \ 1 \leq |x| \leq 2, \ \langle Ax, x \rangle = 0\}$, where A is a symmetric $(d+1) \times (d+1)$ matrix of full rank; note that we *do not* assume anything about the signature of A.

A *k-cone* in $\mathbb{R}^{d+1}$, where $1 \leq k \leq d-1$, is constructed as follows. Let L_0 be a $(d-k+1)$-dimensional linear subspace of $\mathbb{R}^{d+1}$, and let $L_i = L_0 + v_i$ for $i = 1, \ldots, k$, where $v_1, \ldots, v_k$ are linearly independent vectors such that $L_0, v_1, \ldots, v_k$ span $\mathbb{R}^{d+1}$. In each of the subspaces L_i we fix a bounded strictly convex solid F_i such that $E_i = \partial F_i$ is smooth and has nonvanishing Gaussian curvature. Thus E_i is a surface of dimension $d - k$ (i.e. of codimension 1 in

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L_i). We say that a $(k + 1)$-tuple of points $(x_0, x_1, \ldots, x_k)$ is *good* if $x_i \in E_i$, $i = 0, \ldots, k$, and if the outward unit normal vectors to E_i at x_i are the same. We then let

$$S = \bigcup_{(x_0, \ldots, x_k) \ good} \eta(x_0, \ldots, x_k),$$

where $\eta(x_0, \ldots, x_k)$ denotes the convex hull of $x_0, \ldots, x_k$ in $\mathbb{R}^{d+1}$. In Section 7 we prove that this indeed defines a surface of codimension 1 and that each point $a \in S$ belongs to $\eta(x_0, \ldots, x_k)$ for exactly one $(k+1)$-tuple $(x_0, \ldots, x_k)$; we will then call $\eta(x_0, \ldots, x_k)$ the *k-plane* at a, and denote it by $\eta(a)$.

For illustration purposes, consider the simple case when $k = 1$ and L_0 and L_1 are two parallel hyperplanes. If E_0, E_1 are spheres of different radii, then S is a segment of a right circular cone or a slanted circular cone, depending on the relative location of E_0, E_1. Similarly, if E_0, E_1 are spheres of equal radii, S is a segment of a (right or slanted) circular cylinder. However, if E_0, E_1 are randomly chosen ellipsoids, then S will usually not be a cone, a cylinder, or an affine image thereof. For $k \geq 2$, these surfaces may be more difficult to visualize; see Section 7 for more details. It is likely that similar surfaces may be constructed if E_i are allowed to be more general surfaces of codimension 1 in L_i, but since it seems difficult to find the precise conditions on such more general E_i under which the construction works, we choose not to do this here.

What we will actually use in the statement and proof of our main inequality is that these surfaces admit a uniform "plate covering", analogous to that of [18], [5]. Let S be a conical surface or a k-cone, as defined above. For $a \in S$, we use $n(a)$ to denote the unit normal to S at a. We may assume that the map $x \to n(x)$ is continuous; if S is not orientable, we restrict our attention to an orientable subset of S. We always use k to denote the number of "flat" directions of S. For $\delta > 0$, let S_δ denote the δ-neighbourhood of S. We write $A \lesssim B$ if $A \leq cB$ for some constant c independent of δ, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Then S satisfies the following conditions.

Assumption (A). *For each $\delta > 0$, S_δ admits a covering $S_\delta \subset \bigcup_{a \in \mathcal{M}_\delta} \Pi_{a,\delta}$, where $\mathcal{M}_\delta \subset S$, and $\Pi_{a,\delta}$ are finitely overlapping rectangular boxes centered at a with the following properties:*

- $c\Pi_{a,\delta} \subset S_\delta \cap \{x \in \mathbb{R}^{d+1} : (x - a) \cdot n(a) \leq \delta\}$, *for some small c.*

- *All $\Pi_{a,\delta}$ have dimensions $C\delta \times C\delta^{1/2} \times \cdots \times C\delta^{1/2} \times C \times \cdots \times C$, where the short direction is normal to S at a, the long directions are parallel*

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to the k-plane $\eta(a)$ at a, and the mid-length directions are tangent to S at a but perpendicular to $\eta(a)$.

- *(angular separation) For any $a \in \mathcal{M}_\delta$, there are at most $O(1)$ distinct points $b \in \mathcal{M}_\delta$ such that $|n(a) - n(b)| \geq c\delta^{1/2}$.*

- *(consistency) If $0 < \delta \leq \sigma$ and if $b \in \mathcal{M}_\delta \cap \Pi_{a,\sigma}$ for some $a \in \mathcal{M}_\sigma$, then $\Pi_{b,\delta} \subset C''\Pi_{a,\sigma}$.*

Here and in the sequel, "finitely overlapping" means the following: if a family of sets $\mathcal{S}_\delta$ is given for each δ, any $x \in \mathbb{R}^{d+1}$ belongs to at most K sets in $\mathcal{S}_\delta$. $K, C, C', \dots$ denote constants independent of x, a, b, δ, σ, and the choice of $\mathcal{M}_\delta$ and $\mathcal{M}_\sigma$.

We call $\Pi_{a,\delta}$ δ-*sectors* (note that our terminology is slightly different from that of [18], [5]). If S is a nondegenerate hypersurface or a conical surface generated by S_0, we may take $\mathcal{M}_\delta$ to be a maximal $\delta^{1/2}$-separated subset of S or S_0, respectively; the case of k-cones is discussed in Section 7. Whenever the choice of the small parameter δ is clear from the context, we will write Π_a instead of $\Pi_{a,\delta}$. Note that

$$|\Pi_{a,\delta}| \approx \delta^{\frac{d-k}{2}+1}. \tag{3}$$

Let Ξ_a be smooth functions such that $\|\Xi_a\|_1 \approx 1$ and $\{\widehat{\Xi_a}\}_{a \in \mathcal{M}_a}$ is a smooth partition of unity on S_δ with $\text{supp} \widehat{\Xi_a} \subset \Pi_a$. Note that the latter condition implies that $\widehat{\Xi_a} \approx 1$ on a box $c\Pi_a$ of size about $\delta^{\frac{d-k}{2}+1}$, hence $\|\Xi_a\|_2^2 = \|\widehat{\Xi_a}\|_2^2 \approx \delta^{\frac{d-k}{2}+1}$. We may thus choose Ξ_a to have size approximately $\delta^{\frac{d-k}{2}+1}$ on a box dual to Π_a of volume about $\delta^{-\frac{d-k}{2}-1}$, so that the L^1 estimate is satisfied.

If $\text{supp} \hat{f} \subset S_\delta$, we define

$$\|f\|_{p,\delta} = \left(\sum_{a \in \mathcal{M}_a} \|\Xi_a * f\|_p^p \right)^{1/p}$$

for $2 \leq p < \infty$, and

$$\|f\|_{\infty,\delta} = \sup_{a \in \mathcal{M}_a} \|\Xi_a * f\|_\infty.$$

Our main result is the following:

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Theorem 0.1. *Assume that (A) holds. Then for all functions f with $\operatorname{supp} \hat{f} \subset S_\delta$ we have the estimate*

$$\forall \epsilon\, \exists C_\epsilon : \ \|f\|_p \leq C_\epsilon \delta^{-\frac{d-k}{2}+\frac{d-k+1}{p}-\epsilon} \|f\|_{p,\delta}, \tag{4}$$

with C_ϵ depending only on ϵ and on the implicit constants in (A), provided that d, k, p satisfy at least one of the following:

(i) $k < d/3$, $p > p_1(d,k) := 2 + \frac{8}{d-3k}$,

(ii) $k < \frac{3d-3}{4}$, $p > p_2(d,k) := 2 + \frac{32}{3d-4k-3}$.

For the special case of the spherical cone in $\mathbb{R}^{d+1}$, this is the result of [5]. For nondegenerate conical surfaces with $d = 2$ and $k = 1$ (i.e., cones in $\mathbb{R}^3$ generated by plane curves of nonvanishing curvature) the inequality (4) with $p > 74$ (the same exponent as in [18]) has already been obtained by Pramanik and Seeger in [9]. The technique in their paper was to approximate the cone in question by circular cones to which a variant of the result of [18] could be applied. Therefore, the main new cases of interest in the present paper are k-cones with $k \geq 2$ and conical surfaces with both positive and negative curvatures; such surfaces exist in $\mathbb{R}^{d+1}$ only if $d \geq 3$.

Observe that (ii) gives a better range of k for all $d \geq 2$. The range of p given in (i) is better than (ii) if and only if $d > 8k - 3$. The best possible range of p for which (4) could be expected to hold is

$$p \geq 2 + \frac{4}{d-k}. \tag{5}$$

This may be seen by considering the same example as in [18], page 1238: construct a function $f = \sum_{a \in \mathcal{M}_\delta} f_a$, where $\widehat{f_a}$ is supported in a small cube of sidelength δ contained in Π_a, $|f_a| \leq 1$ on a cube Q of sidelength δ^{-1}, f_a is bounded from below on a smaller proportional cube cQ, and decays rapidly outside of Q. Then

$$\|f\|_2^2 \approx \sum \|f_a\|_2^2 \approx M_\delta |Q| = \delta^{-\frac{d-k}{2}} |Q|,$$

$$\|f\|_p^p \gtrsim \|f\|_{L^2(Q)}^p \gtrsim \delta^{-\frac{(d-k)p}{4}} |Q|.$$

Now plug this into the inequality (27) with $\sigma \approx 1$, which will be shown to follow from (4), and take $\alpha \to 0$. Using also (7) and comparing the exponents of δ on both sides, we see that (5) must hold.

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If S is a conical surface in $\mathbb{R}^{d+1}$ generated by a nondegenerate surface S_0 in $\mathbb{R}^d$, the exponent of δ in (4) is $-\frac{d-1}{2}+\frac{d}{p}-\epsilon$ both for S and for S_0. This confirms the observation of [18] that one can deduce (4) for the nondegenerate case from the conical case, for those exponents p for which (4) for the conical case is available. (The idea is to extend a function supported in a d-dimensional neighbourhood of S_0 to a homogeneous degree 0 function defined near S, and then apply (4) to S.) However, going through the entire proof with $k = 0$ allowed, rather than using the shortcut just described, yields a slightly better range of p, namely $p > 2 + \frac{8}{d-1}$ as opposed to $2 + \min(\frac{8}{d-3}, \frac{32}{3d-7})$.

An immediate corollary of Theorem 0.1 is the following result concerning the boundedness of Fourier multipliers associated with S, defined in the usual manner:

$$\widehat{T_\alpha f} = m_\alpha \widehat{f}, \ m_\alpha(\xi) = |\text{dist}\,(\xi, S)|^\alpha \phi(\xi),$$

where $\alpha > 0$ and $\phi \in C_0^\infty$ is a suitable smooth cut-off function supported in a neighbourhood of S.

Corollary 0.2. *Let S be as in Theorem 0.1. Then T_α, defined as above, are bounded on $L^p(\mathbb{R}^{d+1})$ if*

$$\alpha > (d - k + 1)\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}, \tag{6}$$

and if one of the following holds:

(i) $k < d/3$ and either $p > p_1(d, k)$ or $1 \le p < \frac{p_1(d,k)}{p_1(d,k)-1}$,

(ii) $k < \frac{3d-3}{4}$ and either $p > p_2(d, k)$ or $1 \le p < \frac{p_2(d,k)}{p_2(d,k)-1}$,

where $p_i(d, k)$ are as in Theorem 0.1.

The proof for $p > p_i(d, k)$ is identical to that of Corollary 2(ii) in [18], therefore we do not reproduce it here. For $p < p_i(d, k)/(p_i(d, k) - 1)$, the result follows by duality.

The range of α in (6) is sharp for a fixed p, see e.g. [11], pp. 389-390, or [18] for a discussion of the cases $k = 0, 1$. On the other hand, the range of p is not sharp, and in particular it is possible to have L^p-boundedness of T_α for exponents $p > 2$ for which (4) fails. Indeed, the best possible range of p for (4) with $k = 0$ is $p \ge 2 + \frac{4}{d}$ (see (5)), but on the other hand the spherical Bochner-Riesz multipliers with α as in (6) are known to be bounded for $p > 2 + \frac{4}{d+1}$ and $p < 2 - \frac{4}{d+5}$ if $d \ge 2$ [6], and the conjectured range is all $1 \le p \le \infty$. Moreover, in dimension 2 (i.e. $d = 1$, $k = 0$) T_α are known

to be bounded for the optimal range of exponents $4/3 \leq p \leq 4$, $\alpha > 0$, and $\frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha}$, $0 < \alpha \leq 1/2$ ([2], [3], [4]).

For the case $k = 1$, we recover the result of [5] for circular cones and extend it to more general surfaces. We do not know of any earlier results of this type for nontrivial (i.e. not cylindrical) surfaces with $k \geq 2$. For further discussion of the existing literature on cone multipliers, we refer the reader to e.g. [1], [8], [11], [13], [17], [18]. Wolff's inequality (4) for cones has also been used to deduce a variety of other results, including an optimal L^p local smoothing result for solutions of the wave equation [18] and L^p boundedness of maximal operators associated with curves in $\mathbb{R}^3$ [9]. We plan to explore the applications of (4) for more general surfaces in a future paper.

The proof of Theorem 0.1 essentially follows the "induction on scales" arguments in [18], [5], with modifications which we now describe. The inductive argument of [18], [5] (rescaled to our setting) involves an application of (4) on scale $\sqrt{\delta}$, then rescaling each $\sqrt{\delta}$-sector to a neighbourhood of the entire cone via a Lorentz transformation, followed by a second application of (4) on scale $\sqrt{\delta}$. In our more general setting, a non-homogeneous scaling cannot be expected to map S to itself, and in particular Lorentz transformations are usually not available. Instead, we work directly with the two nested decompositions of S. This will require us to prove (4) in somewhat greater generality, allowing for functions with Fourier supports in sectors S_δ^σ, $\delta \leq \sigma \ll 1$. We do this by first dividing the σ-sector in question into ρ-sectors, where $\rho = \sqrt{\sigma\delta}$ is our intermediate scale, and then subdividing the latter into δ-sectors. Thus (4) for S_δ^σ is obtained by combining (4) for S_ρ^σ and S_δ^ρ.

This of course sounds too good to be true, and it indeed is: if implemented exactly as described, the above argument would fail due to the accumulation of the $\delta^{-C\epsilon}$-errors arising at each step of the induction. This problem is resolved as in [18], [5]. Namely, we observe that if the ρ in S_ρ^σ can be replaced by a slightly bigger scale $\rho^{1-\epsilon_0}$ for some fixed ϵ_0, we gain additional factors of $\delta^{C\epsilon_0}$ which absorb the troublesome errors. We then have to find conditions under which it is possible to do so. To this end, we decompose f into standardized "wave packets", Fourier-localized in ρ-sectors and almost localized spatially in the dual plates. The inequality (4), for large p, is a statement about the size of set of large values of f. Fix a tiling of $\mathbb{R}^{d+1}$ by $\rho^{-1+\epsilon_0}$-cubes; then a combinatorial argument, similar in spirit to Bourgain's "bush" argument, shows that if λ is sufficiently large relative to the total number of wave packets, then the sets of wave packets contributing to the

parts of $\{|f| \geq \lambda\}$ localized in different cubes are essentially disjoint. We can now adjust the scale by discarding the non-contributing part of each packet (cf. the "two ends" reduction of [16]). The rest of the proof is arranged so as to make this step possible, and in particular this is what determines our range of p.

1 Notation and preliminaries

General notation:

S: a d-dimensional bounded connected surface in $\mathbb{R}^{d+1}$, C^2 everywhere except boundary, with all curvatures bounded, satisfying (A).

p: an exponent as in Theorem 0.1 which will remain fixed throughout the paper.

$r = 1 - \frac{2}{p} - \frac{2}{p(d-k)}$.

δ, ρ, σ: small dyadic parameters, always satisfying $0 < \delta \leq \rho \leq \sigma \lesssim 1$.

$\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$: small positive numbers with ϵ_{j+1} much smaller than ϵ_j, depending only on d, k, p, to be fixed later. They will remain fixed through Sections 3–6.

t: a dyadic number such that $t \approx (\sigma/\delta)^{\epsilon_0}$.

C, C_i, C', etc.: constants which may depend on the choice of S and p, and in particular on the implicit constants in (A), but will always be independent of δ and of the choice of sector decomposition in (A). They may change from line to line and may be adjusted as needed, in particular after each application of Proposition 3.4.

$A \lesssim B$: $A \leq CB$ for some constant C.

$A \approx B$: $A \lesssim B$ and $B \lesssim A$.

$A \lessapprox B$: $A \lesssim (\log \frac{1}{\delta})^C B$ for some constant C.

χ_E: the indicator function of the set E.

$|E|$: the Lebesgue measure or cardinality of E, depending on the context.

A *logarithmic fraction of E*: a subset of E with measure $\gtrapprox |E|$.

An *l-cube* is a cube of side length l belonging to a suitable fixed l-grid on $\mathbb{R}^{d+1}$; thus any two l-cubes are either identical or have disjoint interiors. If l is fixed, for any $x \in \mathbb{R}^{d+1}$ we let $Q(x)$ be an l-cube such that $x \in Q(x)$.

If R is a rectangular box (e.g. a tube or a plate), we will denote by cR the box obtained from R by dilating it by a factor of c about its center.

If R_0 is a rectangular box centered at the origin, the *dual box* to R_0 is the rectangular box

$$R_0^* = \{x \in \mathbb{R}^{d+1} : \ |x \cdot y| \le 1 \text{ for all } y \in R_0\},$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^{d+1}$. We will sometimes say that two boxes R, R^*, not centered at the origin, are dual to each other if and only if their translates R_0, R_0^* centered at the origin are dual to each other.

We let $\phi(x) = (1+|x|^2)^{-\frac{K}{2}}$ with K large enough, and $\phi_R = \phi \circ u_R^{-1}$, where u_R is an affine map taking the unit cube centered at 0 to the rectangle R; thus ϕ_R is roughly an indicator function of R with "Schwartz tails". If $\mathcal{R}$ is a family of rectangular boxes (usually tubes or plates), we write $\Phi_{\mathcal{R}} = \sum_{R \in \mathcal{R}} \phi_R$.

We let $\psi(x) : \mathbb{R}^{d+1} \to \mathbb{R}$ be a function with the following properties:

(i) $\psi = \eta^2$, where $\hat{\eta}$ is supported in a small ball centered at 0.
(ii) $\psi \ne 0$ on a large cube centered at the origin.
(iii) $\sum_{\nu \in \mathbb{Z}^{d+1}} \psi(x - \nu) \equiv 1$.

We also write $\psi_R = \psi \circ u_R^{-1}$ with u_R as above. If $R = R_\delta$ is the unit cube of sidelength δ centered at 0, we will write $\psi_\delta = \psi_{R_\delta}$.

If a family of functions $\mathcal{F}_\delta$ is given for each δ, we will say that the functions in $\mathcal{F}_\delta$ are *essentially orthogonal* if

$$\| \sum_{f \in \mathcal{F}_\delta} f \|_2^2 \approx \sum_{f \in \mathcal{F}_\delta} \|f\|_2^2.$$

For example, functions with finitely overlapping supports or Fourier supports are essentially orthogonal.

Sector decompositions:

$\Pi_{a,\delta}$: δ-sectors, defined in (A).

$\mathcal{M}_\delta$: set of centers of δ-sectors. For nondegenerate surfaces, we may take $\mathcal{M}_\delta$ to be any $\delta^{1/2}$-separated subset of S; for conical surfaces, $\mathcal{M}_\delta$ may be a $\delta^{1/2}$-separated subset of $S \cap \{|x| = C_0\}$ for some fixed C_0 with $C_1 < C_0 < C_2$, where C_1, C_2 are as in (2). The case of k-cones is discussed in Section 7.

$M_\delta = |\mathcal{M}_\delta|$.

$S_\delta^\sigma(a) = S_\delta \cap \Pi_{a,\sigma}$, for $\delta \le \sigma \lesssim 1$. Whenever the choice of a is unimportant – which will be most of the time – we will write S_δ^σ instead of $S_\delta^\sigma(a)$.

$M_{\sigma,\delta}$: the number of δ-sectors contained in S_δ^σ.

Note that

$$M_\delta \approx \delta^{-(d-k)/2}, \tag{7}$$

$$M_{\sigma,\delta} \approx M_\delta/M_\sigma \approx (\sigma/\delta)^{(d-k)/2}. \tag{8}$$

Indeed, the covering and finite overlap conditions in (A) imply that $M_\delta|\Pi_{a,\delta}| \approx |S_\delta| \approx \delta$, hence (7) follows from (3). Also, by the finite overlap and consistency conditions in (A) we have

$$|S_\delta^\sigma| \approx M_{\sigma,\delta}|\Pi_{a,\delta}| \approx M_{\sigma,\delta}|S_\delta|/M_\delta.$$

In particular, all S_δ^σ have approximately the same size, hence

$$|S_\delta^\sigma| \approx |S_\delta|/M_\sigma.$$

Comparing the last two estimates and using also (7), we get (8).

Dual plates:

We define π_0^a to be the rectangular box dual to $\Pi_a - a$:

$$\pi_0^a = \{x \in \mathbb{R}^{d+1} : \ |x \cdot (y - a)| \leq 1, \ y \in \Pi_a\}.$$

We fix a tiling of $\mathbb{R}^{d+1}$ by translates of π_0^a: $\mathbb{R}^{d+1} = \bigcup_{b \in \Lambda_a} \pi_b^a$, where π_b^a is the translate of π_b^a centered at b. We will refer to π_b^a, $b \in \Lambda_a$, as *plates* dual to the sector Π_a. Thus each $\pi = \pi_b^a$ has a unique sector Π_a to which it is dual; we will sometimes indicate this by writing $\Pi_a = \Pi(\pi_b^a)$.

Note that π^a have dimensions $\approx \delta^{-1}$, $\delta^{-1/2}$, 1 in the directions parallel to the short, medium, and long directions of Π_a, respectively.

2 The wave packet decomposition

Several basic properties of the norm $\|\cdot\|_{p,\delta}$ will be used throughout this paper. We first record the estimate

$$\|f\|_{\infty,\delta} \lesssim \|f\|_\infty \lesssim M_{\sigma,\delta}\|f\|_{\infty,\delta}, \ \operatorname{supp}\widehat{f} \subset S_\delta^\sigma. \tag{9}$$

The first inequality in (9) follows from

$$\|f\|_{\infty,\delta} \leq \max_a \|\Xi_a * f\|_\infty \leq \|\Xi_a\|_1\|f\|_\infty \lesssim \|f\|_\infty,$$

and the second one from the identity $f = \sum_{a \in \mathcal{M}_\delta} \Xi_a * f$. Note also that the functions $\Xi_a * f$ have finitely overlapping Fourier supports and therefore are essentially orthogonal, so that

$$\|f\|_{2,\delta}^2 = \sum_{a \in \mathcal{M}_\delta} \|\Xi_a * f\|_2^2 \approx \|f\|_2^2. \tag{10}$$

Lemma 2.1. *For all $p \geq 2$ we have*

$$\|f\|_{p,\delta} \lesssim \|f\|_2^{2/p} \|f\|_{\infty,\delta}^{1-2/p}. \tag{11}$$

<u>Proof</u> Let $f_a = \Xi_a * f$, then

$$
\begin{aligned}
\|f\|_{p,\delta}^p &= \sum_a \|f_a\|_p^p \leq \max_a \|f_a\|_\infty^{p-2} \cdot \sum_a \|f_a\|_2^2 \\
&= \|f\|_{\infty,\delta}^{p-2} \|f\|_{2,\delta}^2.
\end{aligned}
$$

It now suffices to combine this with (10). $\qquad\square$

We now study the structure of functions with Fourier support in S_δ. More precisely, we want to decompose such functions into "Knapp examples", each of which is Fourier localized in a δ-sector and spatially localized (modulo Schwartz tails) in the plate dual to the sector in question. The definition below and the next two lemmas are identical to the corresponding arguments in [18], [5] modulo notation and rescaling; we include the proofs for completeness.

Definition 2.2. *Let $0 < \delta \leq \sigma \lesssim 1$. We define $\Sigma_\delta^\sigma(a)$ to be the space of all functions of the form $f = \sum_{\pi \in \mathcal{P}} f_\pi$, where $\mathcal{P} = \mathcal{P}(f)$ is a family of δ-plates such that each $\pi_b^{a'} \in \mathcal{P}$ is dual to a sector $\Pi(\pi_b^{a'}) = \Pi_{a'}$ centered at a point $a' \in S_\delta^\sigma(a)$, and*

$$|f_\pi| \lesssim \phi_\pi, \tag{12}$$

$$supp\,\widehat{f_\pi} \subset \Pi(\pi). \tag{13}$$

If $\mathcal{P}' \subset \mathcal{P}$, we say that $f_{\mathcal{P}'} = \sum_{\pi \in \mathcal{P}'} f_\pi$ is a *subfunction* of f. When the choice of a is unimportant, we will often omit a from the notation. Note that of such functions will be denoted by $\Sigma_\delta^\sigma(a)$; as with for functions in Σ_δ^σ we have $\|f\|_\infty \lesssim M_{\sigma,\delta}$ instead of (14). If $\sigma \approx 1$, we will sometimes write $\Sigma_\delta^\sigma = \Sigma_\delta$.

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Lemma 2.3. *Let $f \in \Sigma_\delta^\sigma$. Then $supp\, \widehat{f} \subset S_\delta^\sigma(a)$ and*

$$\|f\|_\infty \lesssim M_{\sigma,\delta}, \tag{14}$$

$$\|f\|_{p,\delta} \lesssim \left(\sum_{\pi \in \mathcal{P}} |\pi| \right)^{1/p} \text{ for } 2 \le p \le \infty. \tag{15}$$

<u>Proof</u> The support statement is clear from the definition. Recall also that all plates π_b^a with fixed a have disjoint interiors; this immediately implies (14). It remains to prove (15). By (11), we only need to do so for $p = 2$ and $p = \infty$.

We first claim that f_π are essentially orthogonal. Indeed,

$$\|f\|_2^2 = \left\| \sum_{a \in \mathcal{M}} \left(\sum_b f_{\pi_b^a} \right) \right\|_2^2 \approx \sum_{a \in \mathcal{M}} \left\| \left(\sum_b f_{\pi_b^a} \right) \right\|_2^2,$$

since $\sum_b f_{\pi_b^a}$ is Fourier supported in Π_a, and the latter have finite overlap. It remains to prove the essential orthogonality of $f_{\pi_b^a}$ for fixed a; this follows from an easy argument using the decay of ϕ_π. Since $\|f_\pi\|_2^2 \lesssim \|\phi_\pi\|_2^2 \approx |\pi|$, this yields (15) for $p = 2$.

To complete the proof for $p = \infty$ and therefore for all p, it suffices to verify that $\|f * \Xi_a\|_\infty \lesssim 1$ for each Ξ_a. We have $f * \Xi_a = \sum_\pi f_\pi * \Xi_a$, where the only non-zero terms are those corresponding to π whose dual plates Π intersect Π_a. Since the number of such Π is bounded, and since the plates π corresponding to each Π are disjoint, it follows that the plates contributing to $f * \Xi_a$ have finite overlap. Hence

$$\|f * \Xi_a\|_\infty \lesssim \max_\pi \|f_\pi * \Xi_a\|_\infty \lesssim \max_\pi \|f_\pi\|_\infty \lesssim 1.$$

$\square$

Lemma 2.4. *Assume that $supp\, \hat{f} \subset S_\delta^\sigma$ and $\|f\|_{\infty,\delta} < \infty$. Then there are $f_\lambda \in \Sigma_\delta^\sigma$, with dyadic λ satisfying*

$$\lambda \lesssim \|f\|_{\infty,\delta} \tag{16}$$

and with corresponding plate families $\mathcal{P}_\lambda$, such that

$$f \approx \sum_\lambda \lambda f_\lambda, \tag{17}$$

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$$\sum_\lambda \lambda^p \sum_{\pi \in \mathcal{P}_\lambda} |\pi| \lesssim \|f\|_{p,\delta}^p \tag{18}$$

for each fixed $p \in [2, \infty)$.

<u>Proof</u> We may assume that $\hat{f}$ is supported in some δ-sector Π_a, so that $\|f\|_{p,\delta} = \|f\|_p$. Let $\psi_b = \psi_{\pi_b^a}$; observe that $\|\psi_b f\|_\infty \lesssim \|f\|_{\infty,\delta} = \|f\|_\infty$. For λ as in (16), we let $\mathcal{P}_\lambda = \{\pi_b^a : \lambda \leq \|\psi_b f\|_\infty \leq 2\lambda\}$ and

$$f_\lambda = \sum_{\pi_b \in \mathcal{P}_\lambda} \lambda^{-1} \psi_b^2 f.$$

To see that $f_\lambda \in \Sigma_\delta^\sigma$, it suffices to verify that each $f_b := \lambda^{-1} \psi_b^2 f$ satisfies (12), (13). Indeed, (12) is immediate from the definition; also, $\hat{f}_b = \widehat{\psi_b^2} * \hat{f}$ and $\widehat{\psi_b^2}$ is supported in a translate of $c\Pi_a$ centered at 0 for some $c \ll 1$, hence (13) follows.

We have $\sum_\lambda \lambda f_\lambda = \sum_j \psi_b^2 f$ and $1 \lesssim \sum_b \psi_b^2 \lesssim \sum_b \psi_b = 1$, so that (17) follows. Moreover, by Bernstein's inequality

$$\lambda^p \approx \|\psi_b f\|_\infty^p \lesssim |\Pi_a| \|\psi_b f\|_p^p \approx |\pi_b^a|^{-1} \|\psi_b f\|_p^p,$$

hence

$$\sum_\lambda \lambda^p \sum_{b:\pi_b^a \in \mathcal{P}_\lambda} |\pi_b| \lesssim \sum_b \|\psi_b f\|_p^p \lesssim \|f\|_p^p$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3   Proof of Theorem 0.1

Let p be as in Theorem 0.1. In the inductive argument that follows, we will need a variant of our main estimate for functions with Fourier support in sectors S_δ^σ. In this setting, our main inequality may be stated as follows: suppose that supp $\hat{f} \subset S_\delta^\sigma$ and

$$\|f\|_{p,\delta} \leq 1, \tag{19}$$

then

$$\|f\|_p^p \lesssim \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\epsilon} \tag{20}$$

for any $\epsilon > 0$, where we recall that

$$r = 1 - \frac{2}{p} - \frac{2}{p(d-k)}. \tag{21}$$

Proposition 3.1. *For any $\epsilon > 0$, we have the estimate*

$$\|f\|_p^p \lesssim \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\epsilon} \|f\|_2^2, \tag{22}$$

for all f with $\|f\|_{\infty,\delta} \lesssim 1$ and supp $\widehat{f} \subset S_\delta^\sigma$.

Assuming Proposition 3.1, let us complete the proof of (20). Pich f with supp $\widehat{f} \subset S_\delta^\sigma$ and obeying (19), then by Lemma 2.4 we have

$$f \approx \sum_{\lambda \leq \|f\|_\infty} \lambda f_\lambda, \; f_\lambda \in \Sigma_\delta^\sigma.$$

From (18) and Lemma 2.3 we have $\|f_\lambda\|_{\infty,\delta} \lesssim 1$ and

$$1 \gtrsim \lambda^p \sum_{\pi \in \mathcal{P}_\lambda} |\pi| \gtrsim \lambda^p \|f_\lambda\|_{2,\delta}^2 \approx \lambda^p \|f_\lambda\|_2^2, \tag{23}$$

which together with (22) yields (20) for λf_λ.

The conclusion follows by summing over λ if we can show that the summation can be restricted to the logarithmically many values of λ in $[\delta^K, \delta^{-K'}]$ for some K, K'. Indeed, for any C we have $\sum_{\lambda \leq \delta^K} \lambda f_\lambda \lesssim \delta^C$ if K is large enough, hence (20) holds for this part. It remains to prove that $\|f\|_\infty \lesssim \delta^{-K'}$. Let $f_a = \Xi_a * f$, then

$$\|f_a\|_\infty \lesssim |\Pi_a|^{1/p} \|f_a\|_p \lesssim |\Pi_a|^{1/p},$$

by Bernstein's inequality and (19). The desired bound follows on summing over a. $\qquad\square$

Proposition 3.1 is an immediate consequence of Proposition 3.3 and Corollary 3.5 below. The main inductive argument is given in Proposition 3.4, the proof of which will occupy Sections 4-6.

Definition 3.2. *We say that $P(p, \alpha, \epsilon)$ holds if for all functions f such that supp $\widehat{f} \subset S_\delta^\sigma$ and $\|f\|_{\infty,\delta} \leq 1$ we have*

$$|\{|f| > \lambda\}| \lesssim \lambda^{-p} \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\alpha} \|f\|_2^2 \,, \tag{24}$$

for all $0 < \delta \leq \sigma \lesssim 1$, provided that δ is small enough.

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Proposition 3.3. $P(p, \alpha, \epsilon)$ *holds for all* $\alpha, \epsilon > 0$.

<u>Proof</u> We will assume that f is as in Definition 3.2. Observe that (24) follows automatically from Chebyshev's inequality if

$$\lambda^{p-2} \lesssim \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\alpha}, \tag{25}$$

By (9), we may assume that $\lambda \leq \|f\|_\infty \lesssim M_{\sigma,\delta}$. Thus for (25) to hold, it suffices if $M_{\sigma,\delta}^{p-2} \lesssim \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\alpha}$, or in other words

$$M_{\sigma,\delta}^{p-2-rp-\alpha} = M_{\sigma,\delta}^{\frac{2}{d-k}-\alpha} \lesssim \delta^{-\epsilon}.$$

This has two consequences of interest to us:

1. (24) holds for all $\alpha > 0$ if $\delta^{-\epsilon} \gtrsim M_{\sigma,\delta}^{\frac{2}{d-k}} \approx \sigma/\delta$ (the last equality comes from (8)),

2. $P(p, \alpha, \epsilon)$ holds for any $\epsilon > 0$ if $\alpha \geq \frac{2}{d-k}$, since then $M_{\sigma,\delta}^{\frac{2}{d-k}-\alpha} \lesssim 1$.

The main inductive step is contained in the following proposition.

Proposition 3.4. *Fix* $p > p_d$ *and* $0 < \epsilon_2 < \epsilon$. *Suppose that* $P(p, \alpha, \epsilon)$ *holds, and let* $\epsilon_0 > 0$ *be sufficiently small (depending only on* p, d, k*). Then (24) holds with* α *and* ϵ *replaced by* $(1 - \frac{\epsilon_0}{6})\alpha$ *and* 4ϵ, *respectively, provided that* $\sigma/\delta \gtrsim \delta^{-\epsilon_2}$.

Assuming Proposition 3.4 for the moment, let us finish the proof of Proposition 3.3. Fix $\alpha, \epsilon > 0$. Fix also $\alpha_0 > 2/(d-k)$, sufficiently small positive numbers ϵ_1, ϵ_2 (depending on d, k, p), and a large integer m_0 so that

$$m_0 > \frac{\log(\alpha/\alpha_0)}{\log(1 - \epsilon_0/6)}, \quad \epsilon_1 < 4^{-m_0}\epsilon, \quad \epsilon_2 < \epsilon_1. \tag{26}$$

By 2. above, $P(p, \alpha_0, \epsilon_1)$ holds.

Assume that we know $P(p, (1 - \epsilon_0/6)^m \alpha_0, 4^m \epsilon_1)$ for some integer $m \geq 0$. We claim that this implies $P(p, (1 - \epsilon_0/6)^{m+1}\alpha_0, 4^{m+1}\epsilon_1)$. Indeed, if

$$\sigma/\delta \lesssim \delta^{-4^{m+1}\epsilon_1},$$

then (24) follows from 1. above. Otherwise, we must have

$$\sigma/\delta \gtrsim \delta^{-4^{m+1}\epsilon_1} \geq \delta^{-\epsilon_1} \geq \delta^{-\epsilon_2},$$

in which case the claim follows from Proposition 3.4. After m_0 iterations, we obtain $P(p, (1 - \epsilon_0/6)^{m_0}\alpha_0, 4^{m_0}\epsilon_1)$. From (26) we have $(1 - \epsilon_0/6)^{m_0}\alpha_0 < \alpha$ and $4^{m_0}\epsilon_1 < \epsilon$, hence $P(p, \alpha, \epsilon)$ follows as required. $\square$

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Corollary 3.5. *Assume that $P(p, \alpha, \epsilon)$ holds. Then for all f with $\operatorname{supp} \hat{f} \subset S_\delta^\sigma$ and $\|f\|_{\infty,\delta} \lesssim 1$ we have*

$$\|f\|_p^p \lesssim \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\alpha} \|f\|_2^2. \tag{27}$$

<u>Proof</u> Write $|f| \approx \sum_\lambda \lambda \chi_{\{|f|\approx\lambda\}}$ with dyadic λ. Then $\|f\|_\infty \lesssim M_{\sigma,\delta}$ by (9); also, (27) is trivial if $\|f\|_\infty \lesssim \delta^n$ for n large enough. The lemma follows by summing (24) over dyadic λ with $\delta^n \lesssim \lambda \lesssim M_{\sigma,\delta}$. $\qquad\square$

4 A substitute for scaling

In this section we develop the geometrical arguments which will replace the scaling arguments of [18], [5]. Instead of rescaling S by powers of δ, we keep S fixed and consider its ρ-neighbourhoods for intermediate values of ρ between δ and 1. We then need a mechanism for efficient conversion of functions with Fourier support in S_δ to functions with Fourier support roughly equal to S_ρ. This is done as follows. Take a function f with $\operatorname{supp} \widehat{f} \subset S_\delta$, and let $f_Q = \psi_Q f$, where Q is a ρ^{-1}-cube. This localizes f spatially in Q, modulo Schwartz tails, and (since $\widehat{f_Q} = \widehat{\psi_Q} * \widehat{f}$) extends its Fourier support to S_ρ. It is instructive to examine what happens to a function $f = f_\pi$ satisfying (12), (13). If $\rho \lesssim \sqrt{\delta}$ (which we will assume in all applications of the lemma), f_Q is essentially obtained from f by shortening the supporting plate π to length about ρ^{-1} in the long direction; to compensate for it, the thickness of the Fourier support Π increases to ρ.

Lemma 4.1. *Assume that $\operatorname{supp} \widehat{f} \subset S_\delta^\sigma$, and let Q be a ρ^{-1}-cube for some $\delta \lesssim \rho \lesssim 1$. Let $f_Q = \psi_Q f$. Then $\operatorname{supp} \widehat{f_Q} \subset S_\rho^\sigma$ and*

$$\|f_Q\|_{p,\rho} \lesssim (\delta/\rho)^{1/p} M_{\rho,\delta}^{1-2/p} \|f\|_2^{2/p} \|f\|_{\infty,\delta}^{1-2/p} \tag{28}$$

for all $p \geq 2$.

<u>Proof</u> Observe that $\widehat{\psi_Q}$ is supported in a ρ-cube centered at 0, and has size about ρ^{-d-1} on a proportional cube. We have $\widehat{f_Q} = \widehat{\psi_Q} * \widehat{f}$. The support statement follows immediately as discussed above. By (11), it suffices to prove (28) for $p = 2$ and $p = \infty$. We have $f_Q = \psi_Q f = \sum_a \psi_Q (\Xi_a * f)$. If we convolute this with a function whose Fourier transform is supported

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in a ρ-sector, the only contributing δ-sectors will be those that intersect the ρ-neighbourhood of the ρ-sector in question, hence the L^∞ bound follows. For $p = 2$, we write

$$\widehat{f_Q}(\xi) = \widehat{\psi_Q} * \widehat{f}(\xi) = \int \widehat{\psi_Q}(\xi - \eta)\chi_{S_\delta}(\eta)\widehat{f}(\eta)d\eta,$$

and use Schur's test. We have $\int \widehat{\psi_Q}(\xi - \eta)\chi_{S_\delta}(\eta)d\eta \lesssim \delta/\rho$ and $\int \widehat{\psi_Q}(\xi - \eta)\chi_{S_\delta}(\eta)d\xi \lesssim 1$. Hence

$$\|f_Q\|_2 = \|\widehat{f_Q}\|_2 \lesssim (\delta/\rho)^{1/2}\|\widehat{f}\|_2 = (\delta/\rho)^{1/2}\|f\|_2 \qquad (29)$$

as required. $\qquad\square$

We also need a substitute for the Lorentz transformations used in [18], [5], [9]. The key geometrical observation turns out to be the following. Suppose that $\widehat{f}$ is supported only in a small part of S_δ, say in S_δ^σ for some $\delta \lesssim \sigma \ll 1$. Compared to S_δ, S_δ^σ is quite flat. We may therefore convolute $\widehat{f}$ with the characteristic function of a box $\mathcal{R}$ whose dimensions in these "flat" directions are larger than ρ, and still stay in a ρ-neighbourhood of S_δ^σ. For $\sigma \ll 1$, this box can be quite a bit larger than a ρ-cube; in the special case when $\sigma = \rho = \delta$, it will be a δ-sector as opposed to a δ-cube. This will result in a considerable gain in (31) below. The reader may be interested to verify that replacing $|R|$ by $|Q| = \rho^{-d-1}$ in (31) would have disastrous consequences at the end of the proof of Theorem 0.1.

Lemma 4.2. *Let $\delta \lesssim \rho \lesssim \sigma \lesssim 1$, and assume that $\sigma \leq \delta^{1/2}$. Let f satisfy $\text{supp}\,\widehat{f} \subset S_\delta^\sigma(a)$. We define $\mathcal{R}$ to be a box centered at 0, with sides parallel to the sides of Π_a and sidelengths $\rho, \rho\sigma^{-1/2}, \rho\sigma^{-1}$ in directions parallel to the short, medium, and long directions of Π_a respectively. We also let R_0 be the box dual to $\mathcal{R}$ centered at 0. Fix a translate R of R_0, and let $f_R = \psi_R f$. Then*

$$|R| = |\mathcal{R}|^{-1} = \rho^{-d-1}\sigma^{(d+k)/2}, \qquad (30)$$

and:

(i) $\text{supp}\,\widehat{f_R} \subset S_\rho^\sigma$,
(ii) f_R obeys (28) with $p = \infty$, i.e. $\|f_R\|_{\infty,\rho} \lesssim M_{\rho,\delta}\|f\|_{\infty,\delta}$,
(iii) if $\rho \lesssim \sqrt{\sigma\delta}$, we have the estimate

$$\|f_R\|_2 \lesssim M_{\sigma,\delta}^{1/2}|R|^{1/2}\|f\|_{\infty,\delta}. \qquad (31)$$

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Note that if $\rho \approx \sqrt{\sigma \delta}$, then R has dimensions $(\sigma \delta)^{-1/2}, \delta^{-1/2}, (\sigma/\delta)^{1/2}$, and (30) becomes

$$|R| \approx \rho^{(k-1)/2} \delta^{-(d+1)/2}. \tag{32}$$

<u>Proof</u> We have $\widehat{f_R} = \widehat{\psi_R} * \widehat{f}$ and supp $\widehat{\psi_R} \subset \mathcal{R}$. Thus the first two parts of the lemma follow by the same argument as in the proof of Lemma 4.1, if we can show that for all $b \in S \cap S_\delta^\sigma(a)$:

$$\mathcal{R} + b \subset S_\rho^\sigma(a) \tag{33}$$

and

$$S_\delta^\rho(b) + \mathcal{R} \subset \Pi_{\rho,a}. \tag{34}$$

Indeed, for any such b we have

$$\Pi_{\sigma,b} \subset C\Pi_{\sigma,a} \subset C'\Pi_{\sigma,b}.$$

It follows that $\Pi_{\sigma,a} + (b-a) \subset C\Pi_{\sigma,b}$, hence $\rho\sigma^{-1}\Pi_{\sigma,a} + (b-a) \subset C\rho\sigma^{-1}\Pi_{\sigma,b}$. But $\rho\sigma^{-1}\Pi_{\sigma,a} - a = \mathcal{R}$, and $C\rho\sigma^{-1}\Pi_{\sigma,b} \subset C'\Pi_{\rho,b}$, by comparing sidelengths and using that $\rho \lesssim \sigma$. Thus

$$\mathcal{R} + b \subset C'\Pi_{\rho,b}, \tag{35}$$

which immediately implies both (33) and (34).

It remains to prove (iii). If $\rho \lesssim \sqrt{\sigma \delta}$, the same argument as in the proof of (35) shows that $b + \mathcal{R}$ is contained in the box $\mathcal{B}_b$ obtained from $C\Pi_{\delta,b}$ by thickening its shortest sidelength to $C\rho$. Hence the functions on the right-hand side of the identity $f_R = \sum_{a \in \mathcal{M}} \psi_R \cdot (\Xi_a * f)$ are essentially orthogonal since their Fourier supports have finite overlap. It follows that

$$\|f_R\|_2^2 \lesssim \sum_a \|\psi_R \cdot (\Xi_a * f)\|_2^2 \lesssim \sum_a \|f\|_{\infty,\delta}^2 \|\psi_R\|_2^2 \lesssim M_{\sigma,\delta}|R| \|f\|_{\infty,\delta}^2$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5   The localization property

Throughout this section we will assume that $\sigma/\delta \gtrsim \delta^{-\epsilon_2}$, where $\epsilon_2 > 0$ was fixed in Section 3.

Definition 5.1. *Let* $f \in \Sigma_\delta^\sigma$. *We say that* f *localizes at* λ *if there are subfunctions* f^Q *of* f, *where* Q *runs over* $t\delta^{-1}$-*cubes (recall that* $t \approx (\sigma/\delta)^{\epsilon_0})$, *such that*

$$\sum_Q |\mathcal{P}(f^Q)| \lesssim |\mathcal{P}(f)| \tag{36}$$

and

$$|\{|f| \geq \lambda\}| \lesssim \sum_Q |Q \cap \{|f^Q| \gtrsim \lambda\}|. \tag{37}$$

Our task is now to find conditions under which f localizes.

Lemma 5.2. *Let* $\mathcal{P}$ *be a family of plates, and let* $W \subset \mathbb{R}^{d+1}$. *Then there is a relation* $\sim$ *between plates in* $\mathcal{P}$ *and* $t\delta^{-1}$-*cubes* Q *such that*

$$|\{Q : \pi \sim Q\}| \lesssim 1 \text{ for all } \pi \in \mathcal{P}, \tag{38}$$

and

$$I_b \lesssim t^{-c_1}|W||\mathcal{P}|^{\frac{1}{2}}, \tag{39}$$

where

$$I_b = \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim Q(x)} \chi_\pi(x) = \sum_{\pi \in \mathcal{P}} |\{x \in W \cap \pi : Q(x) \not\sim \pi\}|.$$

<u>Proof</u> We may assume that $W \subset \{|x| \lesssim \delta^{-1}\}$. For each $\pi \in \mathcal{P}$, we let $Q(\pi)$ be the $t\delta^{-1}$-cube with maximal $|W \cap Q \cap \pi|$ (if there is more than one, pick one arbitrarily). We then say that $\pi \sim Q$ if $Q \cap 10Q(\pi) \neq \emptyset$. Clearly, the number of such cubes (for a fixed π) is $\lesssim 1$. We now prove (39).

By dyadic pigeonholing, there are ν and $\mathcal{P}' \subset \mathcal{P}$ such that $|I_b| \lesssim \nu|\mathcal{P}'|$ and

$$|\{x \in W \cap \pi : \pi \not\sim Q(x)\}| \approx \nu \text{ for each } \pi \in \mathcal{P}'.$$

Thus for each $\pi \in \mathcal{P}'$ there is a cube $Q'(\pi)$ such that $\pi \not\sim Q'(\pi)$ and $|W \cap Q'(\pi) \cap \pi| \gtrsim t\nu$. But then $|W \cap Q(\pi) \cap \pi| \gtrsim t\nu$, by the definition of $Q(\pi)$. The total number of cubes covering W is $\lesssim t^{-d-1}$, hence we may choose Q, Q' so that $Q = Q(\pi)$ and $Q' = Q'(\pi)$ for at least $t^{2d+2}|\mathcal{P}'|$ plates $\pi \in \mathcal{P}'$. Let

$$A = \sum_{\pi \in \mathcal{P}'} |W \cap Q \cap \pi| \cdot |W \cap Q' \cap \pi|,$$

then for Q, Q' as above we have

$$A \gtrsim t^{2d+2}|\mathcal{P}'| \cdot (t\nu)^2 = t^{2d+4}\nu^2|\mathcal{P}'|.$$

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But on the other hand,

$$A = \int_{W \cap Q} \int_{W \cap Q'} \sum_{\pi \in \mathcal{P}'} \chi_\pi(x) \chi_\pi(x') dx' dx$$

$$= \int_{W \cap Q} \int_{W \cap Q'} |\{\pi \in \mathcal{P}' : x, x' \in \pi\}| dx dx'.$$

We claim that the integrand is bounded by t^{-d}. Indeed, let $x \in Q$, $x' \in Q'$, then $|x - x'| \gtrsim t\delta^{-1}$. If $x, x' \in \pi_b^a$. then the angle between $x - x'$ and $n(a)$ is $\lesssim t^{-1}\delta^{1/2}$; but by (A), $|n(a) - n(a')| \geq \delta^{1/2}$ for $a \neq a'$, hence there are at most $\lesssim t^{-d}$ distinct a's as above. Thus

$$A \lesssim t^{-d} |W \cap Q| \cdot |W \cap Q'| \lesssim t^{-d} |W|^2.$$

From this and the lower bound just stated, we have[1] $\nu \lesssim t^{-c} |W| |\mathcal{P}'|^{-1/2}$, so that

$$|I_b| \lesssim \nu |\mathcal{P}'| \lesssim t^{-c} |W| |\mathcal{P}'|^{1/2} \leq t^{-c} |W| |\mathcal{P}|^{1/2}$$

as required. $\square$

In the sequel, we will use a version of Lemma 5.2 with "Schwartz tails". The proof follows word-for-word the standard argument given in detail in [18] and [5], and we see no reason to reproduce it here.

Lemma 5.3. *Let $W \subset \mathbb{R}^{d+1}$ be measurable, and let $\mathcal{P}$ be a family of plates. Fix a large constant M_0. Then, if the constant K in the definition of ϕ has been chosen large enough, there is a relation $\sim$ between $t\delta^{-1}$-cubes Q and plates in $\mathcal{P}$ satisfying (38) and such that*

$$\int_W \Phi_{\mathcal{P}}^b \lesssim t^{-c} |\mathcal{P}|^{\frac{1}{2}} |W| + \delta^{M_0} |W|, \tag{40}$$

where

$$\Phi_{\mathcal{P}}^b(x) = \sum_{\pi \in \mathcal{P}, \pi \nsim Q(x)} \phi_\pi(x) \tag{41}$$

Lemma 5.4. *Let $f \in \Sigma_\delta^\sigma$, with plate family $\mathcal{P}$. Assume that*

$$|\mathcal{P}| \leq t^{2c+2} \lambda^2 \tag{42}$$

with c as in (40). Then f localizes at λ.

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[1] We could keep track of the exact powers of t but will have no need to do so.

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<u>Proof</u> Let $W = \{|f| \geq \lambda\}$, and let $\sim$ be the relation defined in 5.3. For each Q, let $f^Q = \sum_{\pi \sim Q} f_\pi$. By (38), (36) holds. Also, we have

$$\int_W \Phi_\mathcal{P}^b \lesssim t^{-c} |\mathcal{P}|^{\frac{1}{2}} |W| \lesssim t\lambda|W|.$$

Hence we must have $\Phi_\mathcal{P}^b(x) \lesssim t\lambda$ on some set $W^* \subset W$ with proportional measure. Let $x \in W^* \cap Q$, then

$$|f(x) - f^Q(x)| = |\sum_{\pi \not\sim Q} f_\pi(x)| \lesssim \Phi_\mathcal{P}^b(x) \lesssim t\lambda,$$

so that $|f^Q(x)| \gtrsim \lambda$ as claimed. $\qquad\qquad\square$

Lemma 5.5. *Let* $f \in \Sigma_\delta^\sigma(a)$*, with plate family* $\mathcal{P}$*. Fix a tiling* $\{R\}$ *of* $\mathbb{R}^{d+1}$ *by translates of the rectangular box* R_0 *defined in Lemma 4.2, with* $\rho \approx \sqrt{\sigma\delta}$*. Then either* f *localizes at* λ*, or else there is a subfunction* f^* *of* f *such that* $|f^*| \gtrsim \lambda$ *on a logarithmic fraction of* $\{|f| \geq \lambda\}$ *and*

$$\|\psi_R f^*\|_2^2 \lesssim t^{-C} \lambda^{-2} (\sigma/\delta)^{-k/2} M_{\sigma,\delta} |R||\mathcal{P}|. \qquad (43)$$

for each R*.*

<u>Proof</u> For each $\pi = \pi_{b'}^b \in \mathcal{P}$ we let $\tau_{b'}^b$ be a rectangular box containing $\pi_{b'}^b$ with the same dimensions except that all short sidelengths are extended from 1 to $\sqrt{\sigma/\delta}$. From this family of boxes, we choose a maximal subset $\tilde{\mathcal{T}}$ with the property that $\tau' \not\subset C\tau$ for any $\tau', \tau \in \tilde{\mathcal{T}}$ and for a suitable large constant C, and let $\mathcal{T} = \{2C\tau : \tau \in \tilde{\mathcal{T}}\}$. Abusing the notation and standard terminology, we will refer to the boxes in $\mathcal{T}$ as *tubes* and continue to denote them by τ. Each $\pi \in \mathcal{P}$ is then contained in some $\tau(\pi) \in \mathcal{T}$ (if there is more than such τ, we choose one arbitrarily). Observe that the largest angle between two line segments of length $\sim \delta^{-1}$ contained in a tube τ is bounded by $\delta\sqrt{\sigma/\delta} \lesssim \sqrt{\delta}$, hence the angle separation condition in (A) implies that

$$|\{b \in \mathcal{M}_\delta : \pi_{b'}^b \subset \tau \text{ for some } b'\}| \lesssim 1. \qquad (44)$$

We let $\mathcal{T}_m$ be the set of those $\tau \in \mathcal{T}$ with $|\{\pi \in \mathcal{P} : \tau(\pi) = \tau\}|$ between m and $2m$. Let also $\mathcal{P}_m = \{\pi \in \mathcal{P} : \tau(\pi) \in \mathcal{P}_m\}$. Then for some m we have

$$|W| \gtrsim |\{|f| \geq \lambda\}|,$$

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where $W = \{x : |f(x)| \geq \lambda, |f_{\mathcal{P}_m}(x)| \gtrsim \lambda\}$. Fix this value of m, then

$$|\mathcal{T}_m| \lesssim |\mathcal{P}|/m.$$

We now consider two cases.

Case 1: If $\lambda \geq t^{-C}(|\mathcal{P}_m|/m)^{1/2}$, f localizes. The proof is similar to that of Lemma 5.4: we first construct a relation $\sim$ between $t\delta^{-1}$-cubes and tubes in $\mathcal{T}_m$ such that (38) holds, and

$$\int_W \Phi^b_{\mathcal{T}_m} \lesssim t^{-C}|\mathcal{T}_m|^{1/2}|W| + \delta^{M_0}|W|, \tag{45}$$

where M_0 is a large constant and

$$\Phi^b_{\mathcal{T}_m}(x) = \sum_{\tau \in \mathcal{T}_m, \tau \not\sim Q(x)} \phi_\tau(x). \tag{46}$$

The construction is identical to that in Lemma 5.3, therefore we omit it. Since $|\mathcal{T}_m| \lesssim |\mathcal{P}_m| m^{-1}$, it follows that

$$\int_W \Phi^b_{\mathcal{T}_m} \lesssim t^{-C}\Big(\frac{|\mathcal{P}_m|}{m}\Big)^{1/2}|W| \lesssim t\lambda|W|.$$

Hence $\Phi^b_{\mathcal{T}_m} \lesssim t\lambda$ on a subset $W^* \subset W$ with proportional measure. We write $\pi \sim Q$ if $\tau(\pi) \sim Q$, and let

$$f^Q = \sum_{\pi \sim Q} f_\pi.$$

Then (36) follows from (38), and for $x \in W^* \cap Q$ we have

$$|f(x) - f^Q(x)| = |\sum_{\pi \not\sim Q} f_\pi(x)| \lesssim \Phi^b_{\mathcal{T}_m}(x) \lesssim t\lambda,$$

so that $|f^Q| \gtrsim \lambda$ on $W^* \cap Q$ as required. □

Case 2: Assume now that $\lambda \leq t^{-C}(|\mathcal{P}_m|/m)^{1/2}$; we will show that $f_{\mathcal{P}_m}$ satisfies (43). Fix R. It is easy to see (cf. the proof of Lemma 2.3) that $\psi_R f_\pi$ are essentially orthogonal, hence

$$\|\psi_R f_{\mathcal{P}_m}\|_2^2 \approx \|\sum_{\pi \in \mathcal{P}_m} \psi_R f_\pi\|_2^2 \lesssim \int \sum_{\pi \in \mathcal{P}_m} |f_\pi|^2 \phi_R \lesssim \int \Phi_{\mathcal{P}_m} \phi_R,$$

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where at the last step we used (44).

Recall that if $\rho = \sqrt{\sigma/\delta}$ then R has dimensions $(\sigma\delta)^{-1/2}$, $\delta^{-1/2}$, $(\sigma/\delta)^{1/2}$. The main geometrical observation is that if a plate $\pi = \pi_{b'}^b$ intersects CR at all, then a piece of $\tau(\pi)$ of length $\sim (\sigma\delta)^{-1/2}$ is entirely contained in $C'R$ for a suitable choice of C'. Indeed, let $\mathcal{B}_b$ be the box defined in the proof of Lemma 4.2, and recall the inclusion $\mathcal{R} + b \subset \mathcal{B}_b$. Thus the converse inclusion holds for the dual boxes. Hence if π intersects CR, then a piece of π of length $\sim (\sigma\delta)^{-1/2}$ (which is dual to $\mathcal{B}_b$) is entirely contained in $C'R$. Since all dimensions of R are $\gtrsim (\sigma/\delta)^{1/2}$, we may increase C' (if necessary) to obtain the same inclusion for τ.

The corresponding Schwartz tails estimate is

$$\int_R \sum_{\pi:\tau(\pi)=\tau} \phi_\pi \lesssim m(\sigma/\delta)^{-k/2} \int_{C'R} \phi_\tau \qquad (47)$$

for all R and $\tau \in \mathcal{T}$. Namely, let T be the infinite tube extending τ in the direction of its longest axis. If $R \cap CT \neq \emptyset$, (47) follows from the above observation and from the fact that $|\tau| = (\sigma/\delta)^{k/2}|\pi|$. Otherwise, we have the pointwise bound $\sum_{\pi:\tau(\pi)=\tau} \phi_\pi(x) \lesssim m(\sigma/\delta)^{-k/2}\phi_\tau(x)$, which again yields (47).

Summing (47) over all $\tau \in \mathcal{T}$, we obtain

$$\int_R \Phi_{\mathcal{P}_m} \lesssim m(\sigma/\delta)^{-k/2} \int_{C'R} \Phi_{\mathcal{T}}$$

for all R. By an easy covering argument,

$$\int \Phi_{\mathcal{P}_m}\phi_R \lesssim (\sigma/\delta)^{-k/2}m \int \Phi_{\mathcal{T}}\phi_{C'R}.$$

Fix a point x, then the number of tubes with $x \in \tau$ is bounded by $CM_{\sigma,\delta}$. We convert this to a Schwartz tails bound $\Phi_{\mathcal{T}_m} \lesssim M_{\sigma,\delta}$, and deduce that

$$\int \Phi_{\mathcal{P}_m}\phi_R \lesssim m(\sigma/\delta)^{-k/2}M_{\sigma,\delta} \int \phi_{C'R} \lesssim m(\sigma/\delta)^{-k/2}M_{\sigma,\delta}|R|.$$

Recall also that $m \leq t^{-C}|\mathcal{P}_m|\lambda^{-2}$. Therefore

$$\|\psi_\Delta f_{\mathcal{P}_m}\|_2^2 \lesssim \int \Phi_{\mathcal{P}_m}\phi_R \lesssim t^{-C}|\mathcal{P}|\lambda^{-2}(\sigma/\delta)^{-k/2}M_{\sigma,\delta}|R|.$$

$\square$

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6　Proof of Proposition 3.4

In this section we complete the inductive argument. The outline of the proof is as follows. Let $\operatorname{supp}\widehat{f}\subset S_\delta^\sigma$; we want to estimate $\|f\|_p$ in terms of $\|f\|_{p,\delta}$. To this end, we introduce an intermediate scale $\rho=\sqrt{\sigma\delta}$ and first break up S_δ^σ into smaller pieces S_δ^ρ, then decompose these further into δ-sectors. On each S_δ^ρ we apply the inductive assumption $P(p,\alpha,\epsilon)$. We would now like to complete the proof by applying $P(p,\alpha,\epsilon)$ on S_ρ^σ to deal with the coarse decomposition. This, however, would not increase the value of α; therefore at this point we want to change scales from ρ to $\rho^{1-\epsilon_0}$, which will ensure the desired gain. We will see that this is possible if f localizes. Hence the coarse-scale decomposition in Lemma 6.1 is designed so as to allow localization, either on scale ρ or on a second intermediate scale between ρ and σ.

We continue to assume that

$$\sigma/\delta\gtrsim\delta^{-\epsilon_2}, \tag{48}$$

where we recall that ϵ_2 was chosen in Section 3 so that $\epsilon_2<\epsilon$. Fix also a small positive number ϵ_3 with $\epsilon_3<e_2^2$.

Lemma 6.1. *Assume that $P(p,\alpha,\epsilon)$ is known for some p and α. Assume that $\operatorname{supp}\widehat{f}\subset S_\delta^\sigma$ and that $\|f\|_{\infty,\delta}\lesssim 1$. Let $\rho=\sqrt{\sigma\delta}$. Let also $\mathcal{R},R_0$ be the boxes defined in Lemma 4.2, and fix a tiling $\{R\}$ of $\mathbb{R}^{d+1}$ by translates of R_0. Then for any $\lambda\geq\delta^C$ and for any $\epsilon>0$ we may find a λ_* and functions $f_R\in\Sigma_\rho^\sigma$, with respective plate families $\mathcal{P}_R$, such that a logarithmic fraction of $\{|f|\geq\lambda\}$ is contained in $\bigcup_R\{|f_R|\geq\lambda_*\}$ and*

$$\delta^{2\epsilon_3}\lambda M_{\rho,\delta}^{-1}\lesssim\lambda_*\lesssim M_{\sigma,\rho}, \tag{49}$$

$$\sum_{\pi\in\mathcal{P}_R}|\pi|\leq\delta^{-C\epsilon_3}(\frac{\lambda_*}{\lambda})^2\|\psi_R f\|_2^2, \tag{50}$$

$$\sum_R\sum_{\pi\in\mathcal{P}_R}|\pi|\leq(\frac{\lambda_*}{\lambda})^p\delta^{-\epsilon-C\epsilon_3}M_{\rho,\delta}^{rp+\alpha}\|f\|_2^2. \tag{51}$$

　　<u>Proof</u> We write $f=\sum_R\psi_R f$. Using (9) and the Schwartz decay of ψ, it is easy to prove that

$$\{|f|\geq\lambda\}\subset\bigcup_R\{|\psi_R f|\geq c\delta^{\epsilon_3}\lambda\}. \tag{52}$$

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By Lemma 4.2, $\mathrm{supp}\,\widehat{\psi_R f} \subset S_\rho^\sigma$ and $\|\psi_R f\|_{\infty,\rho} \lesssim M_{\rho,\delta}$. Lemma 2.4 now yields a decomposition

$$\psi_R f \approx \sum_h h g_h^R,$$

where $g_h^R \in \Sigma_\rho^\sigma$ and $h \lesssim M_{\rho,\delta}$.

Since we assume that $\lambda \geq \delta^C$, there are logarithmically many relevant dyadic values of h. We may therefore choose $h = h(R)$ so that a logarithmic fraction of $\{|\psi_R f| \geq \delta^{\epsilon_3}\lambda\}$ is contained in the set $\{|h g_h^R| \geq \delta^{2\epsilon_3}\lambda\}$. Finally, we pigeonhole to get a value of h so that a logarithmic fraction of $\{|f| \geq \lambda\}$ is contained in $\bigcup_R\{|h g_h^R| \geq \delta^{2\epsilon_3}\lambda\}$).

Let $\lambda_* = \delta^{2\epsilon_3}\lambda h^{-1}$ and $f_R = g_h^R$, with this value of h. The lower bound in (49) follows from the bound on h just stated; for the upper bound, we use that

$$\lambda_* = \delta^{2\epsilon_3}\lambda h^{-1} \leq \|g_h^R\|_\infty \lesssim M_{\sigma,\rho}.$$

Let $\mathcal{P}_R$ be the plate family for f_R. From (18) we have

$$\sum_{\pi \in \mathcal{P}_R} |\pi| \lesssim h^{-p}\|\psi_R f\|_{p,\rho}^p. \tag{53}$$

Letting $p = 2$ and recalling the definition of λ_*, we deduce (50).

It remains to prove (51). By (53), it suffices to show that

$$\sum_R \|\psi_R f\|_{p,\rho}^p \lesssim \delta^{-C\epsilon_3} M_{\rho,\delta}^{rp+\alpha}\|f\|_2^2. \tag{54}$$

We first claim that

$$\sum_R \|\psi_R f\|_{p,\rho}^p \lesssim \|f\|_{p,\rho} = \sum_b \|\Psi_b * f\|_p^p, \tag{55}$$

where $\{\Psi_b\}$ is the partition of unity defining $\|\cdot\|_{p,\rho}$. Indeed, observe that $\psi_R(\Xi_a * f)$ is Fourier supported in $\Pi_a + \mathcal{R}$. As shown in the proof of Lemma 4.2, the latter set is contained in a $C\rho$-neighbourhood of Π_a. Hence the number of a' such that $\Xi_{a'} * (\psi_R \cdot (\Xi_a * f)) \neq 0$ is bounded by a constant independent of δ and a, and similarly with a and a' interchanged. We thus

have

$$
\begin{aligned}
\sum_R \|\psi_R f\|_{p,\delta}^p &= \sum_R \sum_a \|\Xi_a * (\psi_R f)\|_p^p \\
&\lesssim \sum_b \sum_R \sum_{b'} \|\Xi_a * (\psi_R \cdot (\Xi_{a'} * f))\|_p^p \\
&\lesssim \sum_R \sum_{a'} \|\psi_R \cdot (\Xi_{a'} * f)\|_p^p \\
&\lesssim \sum_{a'} \|\Xi_{b'} * f\|_{p}^p,
\end{aligned}
$$

as required.

Note that $\operatorname{supp} \widehat{\Psi_b * f} \subset S_{C\delta}^\rho$. We also have

$$
\|\Psi_b * f\|_{\infty,\delta} = \max_a \|\Xi_a * \Psi_b * f\|_\infty \lesssim \max_a \|\Psi_b\|_1 \|\Xi_a * f\|_\infty \lesssim \|f\|_{\infty,\delta} \lesssim 1.
$$

Applying the inductive assumption $P(p,\alpha)$ in the form (27) to $\Psi_b * f$, we see that

$$
\|\Psi_b * f\|_p^p \lesssim \delta^{-\epsilon - C\epsilon_3} M_{\rho,\delta}^{rp+\alpha} \|\Psi_b * f\|_2^2.
$$

Combining this with (55) and using the essential orthogonality of $\Psi_b * f$, we obtain (54) as claimed. $\qquad\square$

Lemma 6.2. *Assume that $P(p,\alpha,\epsilon)$ holds, and that $f \in \Sigma_\delta^\sigma$ localizes at λ. Let $\mathcal{P}$ be the plate family for f. Then for any $\beta > (1-\epsilon_0)\alpha$ we have*

$$
|\{|f| > \lambda\}| \lesssim \lambda^{-p} \delta^{-\epsilon} M_{\sigma,\delta}^{rp+\beta} \sum_{\pi \in \mathcal{P}} |\pi|. \tag{56}
$$

<u>Proof</u> Let $W = \{|f| \geq \lambda\}$. The localization assumption means that f has subfunctions f^Q, where Q ranges over $t\delta^{-1}$-cubes, such that (36) holds and

$$
|W| \lessgtr |\bigcup_Q W_Q|,
$$

where $W_Q = Q \cap \{|f_Q| \gtrsim \lambda\}$.

Let $g_Q = \psi_Q f^Q$. By Lemma 4.1, $\operatorname{supp} g_Q \subset S_{\delta/t}$ and $\|g_Q\|_{\infty,\delta/t} \lesssim M_{\delta/t,\delta}$. Applying the inductive hypothesis (24) to $M_{\delta/t,\delta}^{-1} g_Q$, with δ replaced by δ/t and λ replaced by $(\log \frac{1}{\delta})^{-C} M_{\delta/t,\delta}^{-1} \lambda$, we obtain

$$
\begin{aligned}
|\{|g_Q| \gtrsim \lambda\}| &\lessgtr (M_{\delta/t,\delta}^{-1} \lambda)^{-p} (\delta/t)^{-\epsilon} M_{\sigma,\delta/t}^{rp+\alpha} \|M_{\delta/t,\delta}^{-1} g_Q\|_2^2 \\
&\lessgtr \lambda^{-p} M_{\delta/t,\delta}^{p-2} M_{\sigma,\delta/t}^{rp+\alpha} (\delta/t)^{-\epsilon} \|g_Q\|_2^2.
\end{aligned}
$$

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By (29), (15), and (36), we have

$$\sum_Q \|g_Q\|_2^2 \lesssim t \sum_Q \|f^Q\|_2^2 \lesssim t \sum_Q \sum_{\pi \in \mathcal{P}(f^Q)} |\pi| \lessapprox t \sum_{\pi \in \mathcal{P}} |\pi|.$$

Hence

$$
\begin{aligned}
|W| \lessapprox \sum_Q |W_Q| &\lessapprox \sum_Q |\{|g_Q| \gtrapprox \lambda\}| \\
&\lessapprox \lambda^{-p}(\delta/t)^{-\epsilon} M_{\delta/t,\delta}^{p-2} M_{\sigma,\delta/t}^{rp+\alpha} t \sum_{\pi \in \mathcal{P}} |\pi| \\
&\lessapprox \lambda^{-p}(\delta/t)^{-\epsilon} M_{\sigma,\delta}^{rp+\alpha} \sum_{\pi \in \mathcal{P}} |\pi| \cdot t M_{\delta/t,\delta}^{p-2-rp-\alpha},
\end{aligned}
$$

where at the last step we used that $M_{\delta/t,\delta} M_{\sigma,\delta} = M_\delta$. Recall that $p-2-rp = \frac{2}{d-k}$ and $t = (\delta/\sigma)^{\epsilon_0}$. Using also (7), we get

$$t M_{\delta/t,\delta}^{p-2-rp-\alpha} = t M_{\delta/t,\delta}^{\frac{2}{d-k}-\alpha} \approx t \cdot \big(t^{-\frac{d-k}{2}}\big)^{\frac{2}{d-k}-\alpha}$$

$$\approx t^{(d-k)\alpha/2} \approx \big((\frac{\sigma}{\delta})^{-(d-k)/2}\big)^{-\epsilon_0 \alpha} \approx M_{\sigma,\delta}^{-\epsilon_0 \alpha},$$

which yields (56) as required. $\qquad\square$

Lemma 6.3. *Assume that we have $P(p,\alpha,\epsilon)$, and that $f \in \Sigma_\delta^\sigma$ with plate family $\mathcal{P}$ satisfies*

$$|\mathcal{P}| \lesssim t^C \lambda^4 (\delta/\sigma)^{\frac{3d+1}{4}-k}. \tag{57}$$

Then for any $\gamma > (1 - \frac{\epsilon_0}{2})\alpha$ we have

$$|\{|f| > \lambda\}| \lesssim \lambda^{-p} \delta^{-2\epsilon - C\epsilon_3} M_{\sigma,\delta}^{rp+\gamma} \sum_{\pi \in \mathcal{P}} |\pi|. \tag{58}$$

<u>Proof</u> Apply Lemma 5.5 to f. If f localizes at λ, then we are done by Lemma 6.2. Otherwise, pick a subfunction f^* as in Lemma 5.5, and apply Lemma 6.1 to it. We obtain $f_R \in \Sigma_\rho^\sigma$ and a value of λ_* as in (49) so that $|\{|f^*| \gtrapprox \lambda\}| \lessapprox |\bigcup_R \{|f_R| \geq \lambda_*\})|$ and

$$\sum_{\pi \in \mathcal{P}_R} |\pi| \leq \delta^{-C\epsilon_3} (\frac{\lambda_*}{\lambda})^2 \|\psi_R f^*\|_2^2, \tag{59}$$

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$$\sum_R \sum_{\pi \in \mathcal{P}_R} |\pi| \le \delta^{-\epsilon - C\epsilon_3} \left(\frac{\lambda_*}{\lambda}\right)^p M_{\rho,\delta}^{rp+\alpha} \|f^*\|_2^2 \qquad (60)$$

From Lemma 5.5 and (59) we have

$$|\mathcal{P}_R| \rho^{-\frac{d-k}{2}-1} \approx \sum_{\pi \in \mathcal{P}_R} |\pi| \lesssim \delta^{-C\epsilon_3} \frac{\lambda_*^2}{\lambda^4} \left(\frac{\sigma}{\delta}\right)^{-k/2} M_{\sigma,\delta} |R| |\mathcal{P}|.$$

Plugging in (8) and (32), we obtain after some algebra that

$$|\mathcal{P}_R| \lesssim \delta^{-C\epsilon_3} \frac{\lambda_*^2}{\lambda^4} \left(\frac{\sigma}{\delta}\right)^{\frac{3d+1}{4}-k} |\mathcal{P}|.$$

Combining this with (57), we see that

$$|\mathcal{P}_R| \le \delta^{-C\epsilon_3} t^C \lambda_*^2.$$

By Lemma 5.4 and (48), f_R localize at λ_*. Applying Lemma 6.2 we see that

$$|\{|f_R| \ge \lambda_*\}| \lesssim \delta^{-\epsilon} \lambda_*^{-p} M_{\sigma,\rho}^{rp+\beta} \sum_{\pi \in \mathcal{P}_R} |\pi| \qquad (61)$$

for any β with $\beta > (1 - \epsilon_0)\alpha$. Hence

$$
\begin{aligned}
|\{|f| \ge \lambda\}| \;&\lessapprox\; \sum_R |\{|f_R| \ge \lambda_*\}| \\
&\le\; \delta^{-2\epsilon - C\epsilon_3} \lambda_*^{-p} M_{\sigma,\rho}^{rp+\beta} \sum_R \sum_{\pi \in \mathcal{P}_R} |\pi| \\
&\le\; \delta^{-2\epsilon - C\epsilon_3} \lambda^{-p} M_{\sigma,\rho}^{rp+\beta} M_{\rho,\delta}^{rp+\alpha} \|f^*\|_2^2 \\
&\le\; \delta^{-2\epsilon - C\epsilon_3} \lambda^{-p} M_{\sigma,\delta}^{rp+(\alpha+\beta)/2} \sum_{\pi \in \mathcal{P}} |\pi|,
\end{aligned}
$$

where we also used (60) and Lemma 4.1 with $p = 2$. $\qquad \square$

Proof of Proposition 3.4. Assume that supp $\widehat{f} \subset S_\delta^\sigma$ and

$$\|f\|_{\infty,\delta} \le 1. \qquad (62)$$

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We observed at the beginning of the proof of Proposition 3.3 that (24) follows from Chebyshev's inequality if (25) holds. In particular, it holds for any $\alpha, \epsilon > 0$ if $\lambda \lesssim M_{\sigma,\delta}^{rp/(p-2)}$. We may therefore assume that

$$\lambda \geq M_{\sigma,\delta}^{rp/(p-2)}. \tag{63}$$

Recall also that the assumptions of Proposition 3.4 include (48).

Choose $f_R \in \Sigma_\rho^\sigma$ as in in Lemma 6.1, with $\rho = \sqrt{\sigma\delta}$. Suppose that we can prove that f_R obey (58), with δ and λ replaced by ρ and λ_*. Then, using also (51) and (8), we obtain

$$
\begin{aligned}
|\{|f| \geq \lambda\}| \; &\lessapprox \; \sum_R |\{|f_R| \geq \lambda_*\}| \\
&\lesssim \; \lambda_*^{-p} \delta^{-2\epsilon - C\epsilon_3} M_{\sigma,\rho}^{rp+\gamma} \sum_R \sum_{\pi \in \mathcal{P}_R} |\pi| \\
&\lesssim \; \lambda^{-p} \delta^{-3\epsilon - C\epsilon_3} M_{\sigma,\rho}^{rp+\gamma} M_{\rho,\delta}^{rp+\alpha} \|f\|_2^2 \\
&\lesssim \; \lambda^{-p} \delta^{-3\epsilon - C\epsilon_3} M_{\sigma,\delta}^{rp+\theta} \|f\|_2^2
\end{aligned}
$$

for any $\theta > (\alpha + \gamma)/2$, as required.

It thus remain to find conditions under which f_R obey the assumptions of either Lemma 6.2 or 6.3. From (50), (31), (62) we have

$$
\begin{aligned}
|\mathcal{P}_R| \rho^{-\frac{d-k}{2}-1} \; &\approx \; \sum_{\pi \in \mathcal{P}_R} |\pi| \\
&\lesssim \; \delta^{-C\epsilon_3} (\frac{\lambda_*}{\lambda})^2 \|\psi_R f\|_2^2 \\
&\lesssim \; \delta^{-C\epsilon_3} (\frac{\lambda_*}{\lambda})^2 M_{\sigma,\delta} |R|.
\end{aligned}
$$

Plugging in the expression (32) for $|R|$, we obtain after minor simplifications

$$|\mathcal{P}_R| \lesssim \delta^{-C\epsilon_3} (\frac{\lambda_*}{\lambda})^2 M_{\sigma,\delta} (\frac{\sigma}{\delta})^{\frac{d+1}{4}}. \tag{64}$$

Assume first that $d > 3k$ and $p > p_1(d,k) := 2 + \frac{8}{d-3k}$. From (64) and (63) we have

$$|\mathcal{P}_R| \lesssim \delta^{-C\epsilon_3} \lambda_*^2 M_{\sigma,\delta}^{1-\frac{2rp}{p-2}} (\frac{\sigma}{\delta})^{\frac{d+1}{4}}.$$

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Recalling (21), we see that $1 - \frac{2rp}{p-2} = -1 + \frac{4}{(d-k)(p-2)}$. We now also plug in (8) for $M_{\sigma,\delta}$. After some algebra, this yields

$$|\mathcal{P}_R| \lesssim \delta^{-C\epsilon_3} \lambda_*^2 \left(\frac{\sigma}{\delta}\right)^{-\frac{d-3k}{4}+\frac{2}{p-2}}.$$

Recall that $t = (\delta/\sigma)^{\epsilon_0}$. If we assume that ϵ_0 is sufficiently small depending on p, d, k, and if ϵ_3 is small enough compared to the ϵ_2 in (48), then our assumption that $p > p_1(d,k)$ implies that $|\mathcal{P}_R| \lesssim t^C \lambda_*^2$. By Lemma 5.4 f_R localize, hence Lemma 6.2 applies.

Assume now that $p > p_2(d,k) := 2 + \frac{32}{3d-4k-3}$; in this case we will see that (57) holds, with λ and δ replaced by λ_* and ρ. It suffices to check that the right side of (64) is $\lesssim t^C \lambda_*^4 (\rho/\sigma)^{\frac{3d+1}{4}-k}$. After some simplifications, this is equivalent to

$$\lambda_*^2 \lambda^2 \gtrsim \delta^{-C\epsilon_3} t^{-C} M_{\sigma,\delta}\left(\frac{\sigma}{\delta}\right)^{\frac{5d+3-4k}{8}}.$$

By (63) and (49), we have

$$\lambda^2 \lambda_*^2 \gtrsim \delta^{C\epsilon_3} \lambda^4 M_{\rho,\delta}^{-2} \gtrsim \delta^{C\epsilon_3} M_{\sigma,\delta}^{4rp/(p-2)} M_{\rho,\delta}^{-2} \approx \delta^{C\epsilon_3} M_{\sigma,\delta}^{\frac{4rp}{p-2}-1},$$

where at the last step we used (8). It thus suffices to prove that

$$M_{\sigma,\delta}^{\frac{4rp}{p-2}-1} \gtrsim \delta^{-C\epsilon_3} t^{-C} M_{\sigma,\delta}\left(\frac{\sigma}{\delta}\right)^{\frac{5d+3-4k}{8}}.$$

Using again (8) and (21), we see after some algebra that this is equivalent to

$$\left(\frac{\sigma}{\delta}\right)^{\frac{3d-4k-3}{8}-\frac{4}{p-2}} \gtrsim \delta^{-C\epsilon_3} t^{-C}.$$

But if $p > p_2(d,k)$, then the exponent on the left is positive, hence the last estimate follows if ϵ_0 was chosen small enough and if ϵ_3 is small enough compared to ϵ_2 in (48). □

7 Properties of k-cones

We first recall the construction of k-cones described in the introduction. In this section, we will use superscripts to denote Cartesian coordinates in $\mathbb{R}^{d+1}$, e.g. $x = (x^1, \ldots, x^{d+1})$.

Let L_0 be a $(d-k+1)$-dimensional linear subspace of $\mathbb{R}^{d+1}$, and let $L_i = L_0 + v_i$ for $i = 1, \ldots, k$, where $v_1, \ldots, v_k$ are linearly independent

vectors such that $L_0, v_1, \ldots, v_k$ span $\mathbb{R}^{d+1}$. Applying an affine transformation if necessary, we may assume that

$$L_0 = \{(0, \ldots, 0, x^{k+1}, \ldots, x^{d+1}) : x^{k+1}, \ldots, x^{d+1} \in \mathbb{R}\},$$

and that for $i = 1, \ldots, k$,

$$v_i = (v_i^1, \ldots, v_i^{d+1}), \ v_i^i = 1, \ v_i^j = 0 \text{ if } i \neq j.$$

If $x = (x^1, \ldots, x^{d+1}) \in \mathbb{R}^{d+1}$, we will also use the notation $x = (x^\perp, x^\|)$, where $x^\perp = (x^1, \ldots, x^k)$ and $x^\| = (x^{k+1}, \ldots, x^{d+1})$ denote the components orthogonal to L_0 and parallel to it, respectively.

We let $E_i \subset L_i$, $i = 0, \ldots, k$, be surfaces of dimension $d-k$ such that E_i is the boundary of a strictly convex solid in L_i, is smooth, and has nonvanishing Gaussian curvature. Thus for each unit vector $n \in S^{d-k}$, each E_i contains exactly one point x_i such that n is the outward unit normal vector to E_i in L_i at x_i. We will then write $n = n_i(x_i)$. Since E_i is smooth, the mapping $x_i \to n_i(x_i)$ is a smooth diffeomorphism from E_i to S_i.

We say that a $(k+1)$-tuple of points $(x_0, x_1, \ldots, x_k)$, $x_i \in \mathbb{R}^{d+1}$, is *good* if $x_i \in E_i$, $i = 0, \ldots, k$, and if the outward unit normal vectors to E_i in L_i at x_i are the same (i.e. $n_0(x_0) = \cdots = n_k(x_k)$). We then let

$$S = \bigcup_{(x_0, \ldots, x_k) \ good} \eta(x_0, \ldots, x_k),$$

where $\eta(x_0, \ldots, x_k)$ denotes the k-dimensional convex hull of $x_0, \ldots, x_k$ in $\mathbb{R}^{d+1}$.

We first verify that $\eta(x_0, \ldots, x_k)$ is indeed k-dimensional. Indeed, if $\eta(x_0, \ldots, x_k)$ had dimension less than k, then the dimension of the affine space spanned by $\eta(x_0, \ldots, x_k)$ and L_0 would be less than $d+1$. But on the other hand, this affine space contains both L_0 and $v_1, \ldots, v_k$ (since $v_i - x_i \in L_i$), hence must be equal to all of $\mathbb{R}^{d+1}$, which proves our claim.

Let $T_i(x_i)$ be the $d-k$-dimensional affine space tangent to E_i in L_i at x_i. Note that if $(x_0, \ldots, x_k)$ are good, then $T_i(x_i)$, $i = 0, \ldots, k$, are parallel. The above argument shows that $T_i(x_i), n_i(x_i), L_i$ span $\mathbb{R}^{d+1}$. In fact, we can say slightly more. Let D be a closed disc in S^{d-k}, and let

$$E_i|_D = \{x_i \in E_i : n_i(x_i) \in D\},$$

$$S|_D = \bigcup_{(x_0, \ldots, x_k) \ good, \ n_i(x_i) \in D} \eta(x_0, \ldots, x_k).$$

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We may then choose orthonormal bases

$$F^{\parallel}(x_0) = \{u_1(x_0), \ldots, u_{d-k}(x_0)\}, \tag{65}$$

$$F^{\perp}(x_0) = \{u_{d-k+1}(x_0), \ldots, u_d(x_0)\}, \tag{66}$$

for $T_0(x_0)$ and $\eta(x_0, \ldots, x_k)$, respectively, so that each $u_j(x_0)$ depends smoothly on $x_0 \in E_0|_D$. (For $F^{\parallel}$, this is clear from smoothness of S; for $F^{\perp}$, it can be done by applying Gram-Schmidt orthonormalization to $x_1 - x_0, \ldots, x_k - x_0$.) Let $V(x_0)$ denote the volume of a parallelepiped spanned by $F^{\parallel}(x_0)$, $F^{\perp}(x_0)$, $n_0(x_0)$. From the above considerations we have $V(x_0) \neq 0$. But also $V(x_0)$ is a continuous function of x_0, hence

$$V(x_0) \geq c_0 > 0, \ x_0 \in E_0|_D. \tag{67}$$

As noted above, $\eta(x_0, \ldots, x_k)$ depend smoothly on $(x_0, \ldots, x_k)$. Thus to show that S is a smooth surface of codimension 1 in $\mathbb{R}^{d+1}$, it suffices to prove that it is nonsingular, i.e. the different $\eta(x_0, \ldots, x_k)$ do not intersect. This follows from the lemma below.

Lemma 7.1. *Let L be a $(d - k + 1)$-dimensional affine subspace parallel to L_0. If S intersects L, then the cross-section $E := S \cap L$ is a closed smooth $d - k$-dimensional surface with nonvanishing Gaussian curvature, bounding a strictly convex body in L. Moreover, each $\eta(x_0, \ldots, x_k)$ intersects L at a unique point $x \in E \cap \eta(x_0, \ldots, x_k)$. The mapping $x_0 \to x$ is smooth, and the unit outward normal vector to E in L at x is equal to $n_0(x_0)$.*

<u>Proof</u> Suppose that $z = (z^1, \ldots, z^{d+1}) \in L \cap \eta(z_0, \ldots, z_k)$, and that $z \notin \{z_0, \ldots, z_k\}$. Since z belongs to the convex hull of $z_0, \ldots, z_k$, taking the first k coordinates we may write

$$z^{\perp} = \alpha_0(0, \ldots, 0) + \alpha_1(1, 0, \ldots, 0) + \cdots + \alpha_k(0, \ldots, 0, 1),$$

with all $\alpha_i \geq 0$, $\sum_{j=0}^{k} \alpha_j = 1$, and at least one $\alpha_j > 0$. Since along $\eta(z_0, \ldots, z_k)$ each z^j, $j > k$, is a linear function of $z^1, \ldots, z^k$, we must in fact have

$$z = \alpha_0 z_0 + \cdots + \alpha_k z_k.$$

Similarly, if x is any other point in LS with $x \in L \cap \eta(x_0, \ldots, x_k)$, then $x^{\perp} = z^{\perp}$, so that

$$x = \alpha_0 x_0 + \cdots + \alpha_k x_k,$$

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with the same $\alpha_0, \ldots, \alpha_k$. This shows that x depends smoothly on x_0.

Next, we show that the mapping $x_0 \to x$ is one-to-one. Suppose to the contrary that $x \in L \cap \eta(x_0, \ldots, x_k) \cap \eta(y_0, \ldots, y_k)$. Then

$$x^{\|} = \alpha_0 x_0^{\|} + \cdots + \alpha_k x_k^{\|} = \alpha_0 y_0^{\|} + \cdots + \alpha_k y_k^{\|}. \tag{68}$$

Consider, however, the scalar product

$$n_0(x_0) \cdot (\alpha_0 w_0^{\|} + \cdots + \alpha_k w_k^{\|}), \tag{69}$$

where $(w_0, \ldots, w_k)$ ranges over all good k-tuples. By the strict convexity of each E_i, (69) is maximized when $(w_0, \ldots, w_k) = (x_0, \ldots, x_k)$, and only there. But this contradicts the second equality in (68).

Interpreting (69) as the distance from the point $w \in L \cap \eta(w_0, \ldots, w_k)$ to the plane $\{y \in L : n_0(x_0) \cdot (y^{\|} - x^{\|}) = 0\}$, and observing that it is a smooth function of w_0, we see that E is indeed a smooth surface in the neighbourhood of x_0. The argument in the last paragraph now shows that $n_0(x_0)$ is indeed the unit outward normal vector to E in L at x. $\qquad\square$

We now define a covering of S_δ, $\delta > 0$, by δ-sectors as follows. We may assume that S is contained in a large cube of sidelength C_0. Fix a $\delta^{1/2}$-separated subset $\mathcal{N}$ of the unit sphere S^{d-k}. For $i = 0, 1, \ldots, k$, let $\mathcal{M}_i$ be the set of points in E_i where the outward unit normal vectors in L_i belong to $\mathcal{N}$. We then define $\mathcal{M}$ to be the set of the centers of mass of $\eta(x_0, \ldots, x_k)$, $x_i \in \mathcal{M}_i$. We also define the corresponding δ-sectors to be rectangular boxes Π_a centered at $a \in \mathcal{M}$ such that if a is the center of mass of $\eta(x_0, \ldots, x_k)$, then Π_a has sidelengths $C \times \cdots \times C$ in the directions parallel to $\eta(x_0, \ldots, x_k)$, $C\delta^{1/2} \times \cdots \times C\delta^{1/2}$ in directions tangent to S at a but perpendicular to $\eta(x_0, \ldots, x_k)$, and $C\delta$ in the direction parallel to $n(a)$, the normal vector to S at a. Here C is a large constant with $C > 2C_0$, to be fixed later.

First, let $\Pi_a^0 = \Pi_a \cap L_0$. Then Π_a^0 is a rectangular box, centered at x_0, of dimensions at least $C\delta^{1/2} \times \cdots \times C\delta^{1/2} \times C\delta$, where the long axes are tangent to E_0 at x_0. We claim that the dimensions of this box cannot exceed $C'\delta^{1/2} \times \cdots \times C'\delta^{1/2} \times C'\delta$, for some other constant C'. To this end it suffices to prove that the maximal number of disjoint translates of the box spanned by $(\delta n_0(x_0), \delta u_1(x_0), \ldots, \delta u_{d-k}(x_0))$ (recall that u_i were defined in (65)) that can be placed inside Π_a^0 is bounded by $\lesssim \delta^{-(d-k)/2}$. But consider the corresponding translates of the box spanned by $(\delta n_0(x_0), \delta u_1(x_0), \ldots, \delta u_{d-k}(x_0), u_{d-k+1}(x_0), \ldots, u_d(x_0))$. They are also disjoint, each one has volume $\gtrsim \delta^{d-k+1}$ (by (67)), and they are all contained

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in Π_a which has volume $\approx \delta^{\frac{d-k}{2}+1}$. Thus the claim follows by comparing volumes.

This shows that $\{\Pi_a^0\}_{a \in \mathcal{M}_0}$ is, if C is large enough, a standard finitely overlapping covering of the δ-neighbourhood of E_0 in L_0 by rectangular boxes of dimensions roughly $C\delta^{1/2} \times \cdots \times C\delta^{1/2} \times C\delta$, with centers in a $\delta^{1/2}$-separated set $\mathcal{M}_0$. The same argument can now be repeated for $E_1, \ldots, E_k$. In particular, this implies the finite overlap property.

It remains to prove the angular separation property. We may restrict our attention to a small segment $S|_D$ as defined earlier such that $n_0(x_0) \neq -n_0(y_0)$ for any $x_0, y_0 \in E_0|_D$. We then want to prove that if $x, y \in \mathcal{M}$, $x \neq y$, then $|n(x) - n(y)| \gtrsim \delta^{1/2}$. It suffices to prove that

$$|n(x)^{\|} - n(y)^{\|}| \gtrsim \delta^{1/2}.$$

But if $x \in \eta(x_0, \ldots, x_k)$, $y \in \eta(y_0, \ldots, y_k)$, then $n(x)^{\|} = c(x_0)n_0(x_0)$ (since both vectors are orthogonal to $T_0(x_0)$), and similarly $n(y)^{\|} = c(y_0)n_0(y_0)$. By the nonvanishing curvature assumption for E_0, the angle between $n_0(x_0)$ and $n_0(y_0)$ is at least $\delta^{1/2}$; hence it suffices to prove that $c(x_0), c(y_0) \gtrsim 1$. But on the other hand, we have $c(x_0) = V(x_0)$, where $V(x_0)$ is the volume of the parallelepiped defined before (67), and similarly for y_0. Thus the claim follows from (67).

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